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Bounded rationality as a source of loss aversion and optimism: A study of psychological adaptation under incomplete information

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ABSTRACT

We develop a formal model to investigate the implications of bounded rationality for the origin and structure of loss aversion and optimism in marketplaces. Based on Simon's original description, we explicitly model bounded rationality as a decision mechanism that captures incomplete information, psychological adaptation, and rational behavior. We find that the endogenous loss aversion and optimism emerge when the degree of information incompleteness reaches a certain threshold, and both grow to be more prominent when information becomes sparser. Our results highlight that the psychological biases could be expected to take advantage of perceived information incompleteness in terms of value creation.

"*Life is the blended harmony of the yin and yang.*"—Zhuangzi

1. Introduction

Reviewing the recent literature indicates that loss aversion and optimism often appear as two systematic biases of individual investors in attaining their goals (e.g., Hirshleifer, 2001; Daniel et al., 2002; Barberis and Thaler, 2003; Dellavigna, 2009). These psychological biases are understood to be at the root of some robust financial phenomena, but their source mechanism is not yet well reconciled with the standard economic theory. Especially, a coherent way to illustrate the coexistence of the two seemingly mutually exclusive features and to appreciate explicitly how they arise in the first place is still lacking. The purpose of this paper is to propose a mechanism bearing on these issues from the perspective of the "bounded rationality of individuals" (Simon, 1957).

Bounded rationality has vast applications in a wide range of areas, and is also understood in different ways by different people.[1] Here, we choose an adaptive aspect of bounded rationality as the guiding principle of this paper. This principle is highlighted in Simon's own analogy between bounded rationality and a pair of scissors: "*Human rational behavior … is*

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[1] Todd and Gigerenzer (2003) summarize three major interpretations of bounded rationality as optimization under constraints, cognitive illusions, and ecological rationality. The concept adopted in this paper is close to ecological rationality. Also relevant is the adaptive market hypothesis of Lo (2005).

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shaped by a scissors whose two blades are the structure of task environments and the computational capabilities of the actor" (Simon, 1990, P.7). This insight allows us to characterize bounded rationality as a mechanism through which those psychological features are shaped to cope with incomplete information, which arises from the gap between the complexity of the environment where people operate and their limited mental abilities. We refer to the optimal design stance toward human traits as psychological adaptation. The key question about the economic rationale of loss aversion and optimism then becomes: Can psychological adaptation drive individual decision making with incomplete information to occur in a way that reflects the psychological biases within the value-maximization hypothesis?

To address this question, we build a formal model on three key elements. First, investors have incomplete information regarding to the drift parameter (i.e., the expected rate of return) of the stock price process. Second, investors are completely "Bayesian" rational. They are both Bayesian and inter-temporally optimizers in maximizing subjective satisfaction. We construct their rational behavior by deriving explicit forms for the solutions of a dynamic allocation problem where investors are prone to preference bias (loss aversion), belief bias (optimism or pessimism), and incomplete information (parameter uncertainty). Third, psychological adaptation endows investors with the desired traits of loss aversion and optimism (pessimism) in terms of value maximization. Specifically, the psychological biases, if any, are endogenized in our model as the solutions of maximizing the objective expected growth rate of wealth accumulation. The solutions allow us to evaluate whether psychological adaptation plus incomplete information lead to systematic biases. By comparing the optimal belief and preference features of investors with different degrees of information incompleteness, we also see the causal relationship between incomplete information and the two psychological biases.

Our results show that endogenous loss aversion and optimism will emerge once the level of information incompleteness reaches a certain threshold. More specifically, there are two regions: the "simple" region where information is rather complete, and the "complex" region where information is rather incomplete. In the simple region, loss aversion and optimism cannot emerge as the optimal attitudes. In the complex region, some appropriate alignments of loss aversion and optimism are beneficial for making good decisions, and hence both arise in the optimal attitudes. In this region, it is also clearly shown that such endogenous biases will increase with the degree of information incompleteness.

Although incomplete information has similar implications for both loss aversion and optimism in our model, the relevant mechanisms are essentially different. Loss aversion itself, without the help of optimism, can benefit investors when information is rather incomplete. The basic reason lies in the way that loss aversion makes investors more cautious to take unrewarded risk due to information incompleteness. However, in our model, a belief bias (optimism or pessimism) itself is always a disadvantage, which is in accordance with a series of recent theoretical studies (Sandroni, 2000; Blume and Easley, 2006; Yan, 2008). But optimism can achieve a certain efficiency in its coordination with loss aversion as it can counterbalance the effect of loss aversion that leads to less allocations to stocks. As such, information incompleteness leaves "rooms" for loss aversion, and loss aversion makes "rooms" for optimism.

The contribution of this paper is thus to formalize the bounded rationality mechanism in an adaptive form, to demonstrate how it can reconcile loss aversion and optimism with the value-maximization hypothesis, and to derive a general relationship between incomplete information and the psychological biases. The explicit modeling of investor psychology helps us for the better understanding of the complexity of systematic psychological biases from standard rationality and how they arise in the first place. This also suggests several applications. For example, it offers an immediate justification for the premises of loss aversion and optimism in financial studies, e.g., Benartzi and Thaler (1995).

Our paper relates to the literature on natural or market selection. Our model accepts the key results that have been formally identified by the market-selection analysis in Blume and Easley (1992, 2006), the maximization of expected wealth growth rate and the Bayesian learning, as the basic assumptions. Accordingly, the selection pressure (e.g., from market power and/or evolution) can be regarded as a good reason that individuals behave as if they are endowed with psychological adaptation mechanisms about loss aversion and optimism, and our bounded-rationality research program can be viewed as an evolutionarily informed framework. On a technical note, however, our work is not a general survival analysis. Our functional argument about loss aversion and optimism primarily is of an adaptive, rather than survival, nature. In fact, the two approaches are complementary, but suited to different questions. Our results, although do not directly guarantee that loss-averse and optimistic agents can survive in the long run, immediately provide a microfoundation for the argument that bounded rationality serves as a source of loss aversion and optimism.

Our paper is closely related to the literature on parameter uncertainty and learning in financial markets. A number of studies have investigated the implications of parameter uncertainty and learning on various investment problems (see, Pástor and Veronesi, 2009). Our paper incorporates reference-dependent subjective expected utility into the works of Lakner (1995, 1998), and hence also belongs to the literature on dynamic problems with reference-related objective functions, e.g., Basak (1995), Carpenter (2000), Basak and Shapiro (2001), Berkelaar et al. (2004). However, none of these papers investigates the dynamic portfolio choice problem that we specifically address in this paper, that is, the problem that accounts for both nonstandard subjective expected utility and parameter uncertainty.

Our paper proceeds as follows. We introduce in Section 2 the continuous-time model we adopt for incorporating incomplete information, loss aversion, and optimism. Then we present in Section 3 the optimal solutions of the model, and offer some comparative statics analysis of how information incompleteness, preference bias, and belief bias determine investor behavior. In Section 4, we analyze the relationships among loss aversion, optimism, incomplete information, and other components in our framework. In Section 5, we discuss the model's predictions and their implications. We finally conclude our paper in Section 6.

2. Model framework

We consider a continuous-time financial market consisting of a riskless bond with a constant interest rate, r_f, and a risky stock, whose price $S(t)$ satisfies

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dB(t), \quad S(0) = S_0, \tag{1}$$

where μ and $\sigma > 0$ are constants and $B(t)$ is the standard Brownian motion. In the continuous-time setting, Merton (1980) has demonstrated that the optimal estimator of the volatility (or variance) converges to the true value, while the estimator of the expected return is not. Then, following the literature on incomplete information (see, e.g., Gennotte, 1986; Brennan, 1998; Lakner, 1998; Rogers, 2001), we assume that investors know r_f and σ exactly, but do *not* know the drift parameter μ exactly, instead, having a normal *prior* belief on μ at time zero,

$$\tilde{\mu} \sim N(m_0, v_0),$$

where $N(m_0, v_0)$ denotes the normal distribution function with mean m_0 and variance v_0. As the variance of the mean estimation cannot exceed the population variance, we assume $v_0 \le \sigma^2$.

In our model, investors can observe the stock price, and update their beliefs on μ in a Bayesian way as the stock price process evolves. Denote the investor's filtration generated by the stock price process as $\{\mathcal{F}_t^S = v(S(s), 0 \le s \le t)\}$. Accordingly, the conditional mean of the drift parameter is $m(t) = \mathrm{E}[\mu | \mathcal{F}_t^S]$, and the conditional mean of the market price of risk $(\theta = (\mu - r_f)/\sigma)$ is

$$\theta_m(t) = \mathrm{E}[\theta | \mathcal{F}_t^S] = \frac{m(t) - r_f}{\sigma}.$$

As μ is unknown, investors are unable to observe the noise term $B(t)$ from Eq. (1). Rather, they are able to deduce the following related processes:

$$\tilde{B}(t) = \theta t + B(t) \quad \text{and} \quad \overline{B}(t) = \tilde{B}(t) - \int_0^t \theta_m(u)\, du,$$

where $\tilde{B}(t)$ is the risk-neutral Brownian motion, and $\overline{B}(t)$ is a standard Brownian motion with respect to the price filtration $\mathcal{F}_t^S$ (Liptser and Shiryayev, 1977).

From the theory of the classical Kalman–Bucy filter (Liptser and Shiryayev, 1977), we know that the conditional expectation $\theta_m(t)$ satisfies

$$d\theta_m(t) = \gamma(t)(d\tilde{B}(t) - \theta_m(t)dt), \quad \theta_m(0) = \theta_0 = \frac{m_0 - r_f}{\sigma}, \tag{2}$$

and the conditional variance $\gamma(t) = \mathrm{E}[(\theta - \theta_m(t))^2 | \mathcal{F}_t^S]$ is determined by the following Riccati equation:

$$\frac{d\gamma(t)}{dt} = -\gamma^2(t), \quad \gamma(0) = \gamma_0 = \frac{v_0}{\sigma^2},$$

which leads to its solution as follows:

$$\gamma(t) = \frac{\gamma_0}{\gamma_0 t + 1} \quad \text{and} \quad \theta_m(t) = \frac{\theta_0 + \gamma_0 \tilde{B}(t)}{\gamma_0 t + 1}. \tag{3}$$

Let $W(t)$ be the wealth of the investor at time t and $\pi(t)$ be the fraction of the wealth invested in the risky stock at time t. Then, the corresponding wealth process is governed by

$$\begin{aligned} dW(t) &= W(t)\{(\pi(t)(\mu - r_f) + r_f)dt + \pi(t)\sigma\, dB(t)\} \\ &= W(t)\{(\pi(t)(m(t) - r_f) + r_f)dt + \pi(t)\sigma\, d\overline{B}(t)\}, \end{aligned} \tag{4}$$

with $W(0) = W_0 > 0$ being the initial wealth of the investor.

As suggested by the phrase "survival of the richest", we take the growth rate of wealth accumulation, i.e., the continuously compounded (log) return of the investment

$$R(T) = \ln \frac{W(T)}{W_0},$$

as the measure of the investor's relative success (i.e., the term "richest") in the market. Although the wealth growth rate has also been defined as $R(T)/T$ in some other studies, the two definitions are clearly equivalent as T is given as an exogenous variable in this study. As known from the literature (e.g., Blume and Easley, 1992; Hakansson and Ziemba, 1995; Robson, 2002), the expected log-return maximization, sometimes also referred to as the *Kelly criterion* (Kelly, 1956), is an appropriate criterion for evolutionary success in the conventional asset market setting, because its strategy generates almost surely the greatest wealth among all strategies in the long run. Hence, we formulate the adaptiveness criterion as

$$\max \quad \mathrm{E}[R(T)]. \tag{P1}$$

Note that the expectation $\mathrm{E}[\cdot]$ is based on the *actual* distribution of the stock price process, where incomplete information is of irrelevance.

Following the common practice of building theories in behavioral economics, we retain the basic architecture of the standard model (P1) and add to it assumptions about incomplete information, loss aversion, and optimism to formulate investor behavior. We use the relative imprecision of the prior belief,

$$\gamma_0 = \nu_0/\sigma^2 \in [0,1],$$

to measure the level of information incompleteness. As $\gamma(t)$ is a monotonically increasing function of γ_0, the level of γ_0 reflects the incomplete extent of the investor's perceived information at any time t. In our model, we treat γ_0 as an exogenous variable.

Besides incomplete information, the investor is also prone to the belief and preference biases (if any). To measure belief bias, we adopt the ratio of the systematic component of expectation error to the true parameter:

$$b = \frac{m_0 - \mu}{\mu}. \tag{5}$$

Hence, the cases with $b > 0$ imply optimistically biased beliefs.

For the preference bias, we adopt the value function of the following form[2]:

$$u(x,\lambda) = \begin{cases} x - \overline{x}, & x \geq \overline{x}, \\ \lambda(x - \overline{x}), & x < \overline{x}. \end{cases}$$

where x is the investment return, $\overline{x}$ is the reference level, and $\lambda \geq 1$ is the coefficient of preference bias (loss aversion). When $\lambda > 1$, investors hold loss-averse preferences.

The investor's subjective value function is then constructed as

$$u(R(T),\lambda),$$

where the reference return $\overline{x}$ is specified as $\overline{R} = \ln(\overline{W}/W_0)$ with $\overline{W}$ being the corresponding reference wealth. In brief, investors behave optimally in the sense of maximizing their subjective expected satisfaction,

$$\max \quad \mathrm{E}^s[u(R(T),\lambda)], \tag{P2}$$

where $\mathrm{E}^s[\cdot]$ denotes the *subjective* expectation based on the prior belief (m_0,ν_0), hence (b,γ_0), and the stock price process, i.e., $\mathrm{E}[\cdot|\mathcal{F}_0^S]$.

The question now becomes how the belief and preference biases (b and λ) are determined? According to the adaptiveness criterion (P1), we can then express the endogenous psychological biases within our bounded rationality program as the solution of the following maximization problem:

$$\max_{\lambda,b} \mathrm{E}[R(T|\lambda,b)], \quad \text{s.t. } R(T|\lambda,b) \text{ is the return of Problem (P2)}. \tag{P1$'$}$$

This problem can be viewed as an attempt to achieve a match between the internal mind, the problem (P2), and the external market, the problem (P1).

When the investor's information is complete (i.e., μ is known), the adaptation problem has a simple solution: $\lambda = 1$, $b=0$. That said, loss aversion and optimism are not required. When information incompleteness is inherent in the decision making process of the investor, however, this perfect alignment ($\lambda = 1$, $b=0$) is not necessarily optimal in the adaptiveness sense: loss aversion and optimism may actually be beneficial to the extent to which they help to enhance the actual expected wealth growth rate. In this case, the adaptation problem (P1′) then becomes crucial. Simon (1986) has addressed this issue well:

> "If, on the other hand, we accept the proposition that both the knowledge and the computational power of the decision maker are severely limited, then we must distinguish between the real world and the actor's perception of it and reasoning about it. ... Our theory must include not only the reasoning processes but also the processes that generate the actor's subjective representation of the decision problem, his or her frame." (Simon, 1986, P. 211)

The actual process by which mental devices design "the actor's subjective representation of the decision problem" can be quite complicated, but that is not our concern. Here, we focus on how loss aversion and optimism are associated with information incompleteness. The adaptation problem (P1′) allows us to make meaningful guesses about those associations that human mind embodies in an elegant and parsimonious way (Cosmides and Tooby, 1994). Recently, neuroscience studies (Sharot et al., 2007; Tom et al., 2007) demonstrate that the underlying mechanisms of loss aversion and optimism may be biological in nature. At the biological level, one possible explanation is thus that such associations are designed by natural selection to solve the recurrent intertemporal economic tradeoff over the history of human being (Robson, 2002) and remain functional to solve today's trading problems.

[2] This functional form captures the loss averse feature, and omits other potential preference features, such as risk-aversion (risk-seeking) over gains (losses) of the prospect theory formulated by Kahneman and Tversky (1979). We use this rather simple preference structure to focus more directly on loss aversion in our analysis. This restrictive setting is also supported by the observation of Benartzi and Thaler (1995).

Before turning into the application, there are several points worth noting. First, this paper employs a finite-horizon partial-equilibrium setting, in sharply contrast with the infinite-horizon general-equilibrium setting in the literature on market selection (e.g., Sandroni, 2000; Blume and Easley, 2006; Yan, 2008). In doing so, as stated in the Introduction, the paper is primarily on adaptation, which might be expected in the short run (Robson, 2002), rather than market selection, which requires a long time to become manifest (Yan, 2008). As such, the bounded-rationality approach in this paper is helpful for shedding new light on how those behavioral regularities occur in the context of investment decisions.

Second, the price taking setting also distinguishes our model from the noise-trader models with imperfectly competitive markets pioneered by De Long et al. (1990, 1991). In such noise-trader models, the mechanism that the demands due to investor sentiment affect price levels plays a central role in understanding the prevalence and persistence of those biases, which is not involved in this paper. Meanwhile, it is difficult for this class of models to explain why loss aversion and optimism arise in the first place.[3]

Third, our development of bounded rationality is not bounded by cognitive illusion or self-deception (see, e.g., Hirshleifer, 2001). Thus, loss aversion and optimism in this paper should not be viewed as the impulsive emotions simply due to hedonic feelings (e.g., Brunnermeier and Parker, 2005). Rather, these features are economically plausible because our adaptation channel through which information incompleteness produces loss aversion and optimism is its influence on value creation. In this sense, our results support the engagement of loss aversion and optimism in deliberation.

In fact, it is impossible to be fully comprehensive on the rich implications of bounded rationality and the potential sources of psychological biases. Our focus is how loss aversion and optimism, that go to make up investor behavior, could arise and change as the optimal cognitive responses of individual investors with respect to their incomplete-information bounds. The contribution of this paper is formalized this mechanism in terms of the basic market rule, value or profit maximization.

3. Optimization under preference bias, belief bias, and incomplete information

3.1. Investor's optimization

Similar to Lakner (1995, 1998), we apply the martingale method to restate the investor's dynamic optimization problem in (P2) as the following static variational problem:

$$\max_{W(T)\geq 0} \quad \mathrm{E}^s[U(W(T))]$$
$$\text{s.t.} \quad \mathrm{E}^s[\zeta(T)W(T)] = W_0, \tag{P2$'$}$$

where the state-price density process $\zeta(t)$ satisfies

$$\zeta(t) = \exp\left\{-r_f t - \int_0^t \theta_m(u)\, d\tilde{B}(u) + \frac{1}{2}\int_0^t \theta_m(u)^2\, du\right\}, \quad \zeta(0) = 1. \tag{6}$$

By Itô formula, we can further derive the exact analytical expression of the state-price density process as

$$\zeta(t) = g(t)\exp\left\{-\frac{1}{2}\frac{\gamma_0 t + 1}{\gamma_0}\theta_m(t)^2\right\}, \tag{7}$$

where $g(t) = \exp\{-r_f t + \frac{1}{2}\ln(\gamma_0 t + 1) + \frac{1}{2}(m_0 - r_f)^2/\nu_0\}$. It is clear that the state-price density process $\zeta(t)$ is bounded within $[0, g(t)]$.

The following proposition characterizes the optimal terminal wealth of the investor, while its proof can be found in the Appendix.

Proposition 1. *The investor's optimal terminal wealth is*

$$W(T) = \begin{cases} \dfrac{1}{y\zeta(T)} & \text{if } \zeta(T) < \underline{\zeta}, \\ \overline{W} & \text{if } \underline{\zeta} \leq \zeta(T) \leq \overline{\zeta}, \\ \dfrac{\lambda}{y\zeta(T)} & \text{if } \zeta(T) > \overline{\zeta}, \end{cases} \tag{8}$$

where $\underline{\zeta} = 1/(y\overline{W}), \overline{\zeta} = \lambda/(y\overline{W})$ *and* $y \geq 0$ *is a constant satisfying* $\mathrm{E}^s[W(T)\zeta(T)] = W_0$.

In order to obtain the solution for the adaptation problem in (P1$'$), we need to consider the actual expected log-return of the investor examined in Proposition 1. Without loss of generality, we normalize the initial wealth at $W_0 = 1$. Then, by some calculations, we can obtain the following corollary with its proof being given in the Appendix.

[3] In addition, the class of imperfectly competitive models uses arithmetic return to justify their mechanisms, while our model uses log return, which is consistent with utilities.

Corollary 1. *For given λ and b, the adaptiveness criterion in* (P1′) *under the optimal terminal wealth derived from Problem* (P2) *is*

$$
\begin{aligned}
\mathrm{E}[R(T|\lambda,b)] &= \mathrm{E}[R(T|1,b)]+\mathrm{E}[R(T|\lambda,b)-R(T|1,b)] \\
&= -\ln g(T)+\frac{1}{2}\frac{\gamma_0 T+1}{\gamma_0}(\mu_\theta^2+\sigma_\theta^2)+(\ln\lambda-\ln y)H(x_1)-\ln y[1-H(x_2)] \\
&\quad +\ln\overline{W}[H(x_2)-H(x_1)]+\ln g(T)[H(x_2)-H(x_1)] \\
&\quad -\frac{1}{2}\frac{\gamma_0 T+1}{\gamma_0}[G(\sqrt{x_2})-G(\sqrt{x_1})+G(-\sqrt{x_1})-G(-\sqrt{x_2})],
\end{aligned}
$$

where

$$
\mu_\theta=\frac{\theta_0+\gamma_0\theta T}{\gamma_0 T+1},\quad \sigma_\theta=\frac{\gamma_0\sqrt{T}}{\gamma_0 T+1},\quad x_1=\frac{-\ln\overline{\zeta}+\ln g(T)}{\frac{1}{2}\frac{\gamma_0 T+1}{\gamma_0}},\quad x_2=\frac{-\ln\underline{\zeta}+\ln g(T)}{\frac{1}{2}\frac{\gamma_0 T+1}{\gamma_0}}
$$

$$
H(x)=\Phi\left(\frac{\sqrt{x}-\mu_\theta}{\sigma_\theta}\right)-\Phi\left(\frac{-\sqrt{x}-\mu_\theta}{\sigma_\theta}\right),
$$

$$
G(x)=\Phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right)\mu_\theta^2-2\mu_\theta\sigma_\theta\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right)+\sigma_\theta^2\left[\Phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right)-\frac{x-\mu_\theta}{\sigma_\theta}\phi\left(\frac{x-\mu_\theta}{\sigma_\theta}\right)\right],
$$

and $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and the cumulative distribution function of the standard normal distribution, respectively.

This corollary reveals the explicit relationships between the return outcome and the psychological determinants of the investor's rational behavior in our framework, which allows us to precisely obtain the optimal values of λ and/or b as the solution of the adaptation problem (P1′). As a preliminary exercise, it is useful to visit the adaptation problem (P1′) when only belief bias alone is present. When there is no preference bias, i.e., $\lambda=1$, we have

$$
\mathrm{E}[R(T|1,b)]=-\ln g(T)+\frac{1}{2}\frac{\gamma_0 T+1}{\gamma_0}(\mu_\theta^2+\sigma_\theta^2).
$$

When $\lambda=1$, the solution of the problem (P1′), b^*, must satisfy

$$
\frac{\partial\mathrm{E}[R(T|1,b)]}{\partial b}=\frac{\theta}{\gamma_0}(\mu_\theta-\theta_0)=0.
$$

Some simple algebras show that $b^*=0$ (i.e., $\theta_0=\theta$), which implies that optimism, or pessimism, should not arise on the beliefs of loss-neutral investors. The result is consistent with Sandroni (2000), Blume and Easley (2006) and Yan (2008).

3.2. Properties of investor's optimal behavior

We present in the following proposition explicit expressions for the investor's optimal wealth and portfolio strategies during the investment horizon, while placing its proof in the Appendix.

Proposition 2. *For Problem* (P2), *the investor's time-t optimal wealth is given by*

$$
\begin{aligned}
W(t) &= \frac{1}{y\zeta(t)}\left[\Phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right)+1-\Phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right]+\overline{W}e^{-r_f(T-t)}\left[\Phi\left(\frac{d_3(\overline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)-\Phi\left(\frac{d_3(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)\right. \\
&\quad \left.+\Phi\left(\frac{d_4(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)-\Phi\left(\frac{d_4(\overline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)\right]+\frac{\lambda}{y\zeta(t)}\left[\Phi\left(\frac{d_2(\overline{\zeta})}{d(t)}\right)-\Phi\left(\frac{d_1(\overline{\zeta})}{d(t)}\right)\right],
\end{aligned}
$$

and the optimal fraction of his wealth invested in the stock is

$$
\begin{aligned}
\pi(t) &= \frac{1}{W(t)\sigma}\left\{\frac{\theta_m(t)}{y\zeta(t)}\left[\Phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right)+1-\Phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right]-\frac{1}{yd(t)\zeta(t)}\frac{\gamma_0}{\gamma_0 t+1}\left[\phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right)-\phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right]\right. \\
&\quad -\frac{\overline{W}e^{-r_f(T-t)}}{\sqrt{T-t}}\left[\phi\left(\frac{d_3(\overline{\zeta})}{\overline{d}(t)}\right)-\phi\left(\frac{d_3(\underline{\zeta})}{\overline{d}(t)}\right)+\phi\left(\frac{d_4(\underline{\zeta})}{\overline{d}(t)}\right)-\phi\left(\frac{d_4(\overline{\zeta})}{\overline{d}(t)}\right)\right]+\theta_m(t)\frac{\lambda}{y\zeta(t)}\left[\Phi\left(\frac{d_2(\overline{\zeta})}{d(t)}\right)-\Phi\left(\frac{d_1(\overline{\zeta})}{d(t)}\right)\right] \\
&\quad \left.-\frac{\lambda}{yd(t)\zeta(t)}\frac{\gamma_0}{\gamma_0 t+1}\left[\phi\left(\frac{d_2(\overline{\zeta})}{d(t)}\right)-\phi\left(\frac{d_1(\overline{\zeta})}{d(t)}\right)\right]\right\},
\end{aligned}
$$

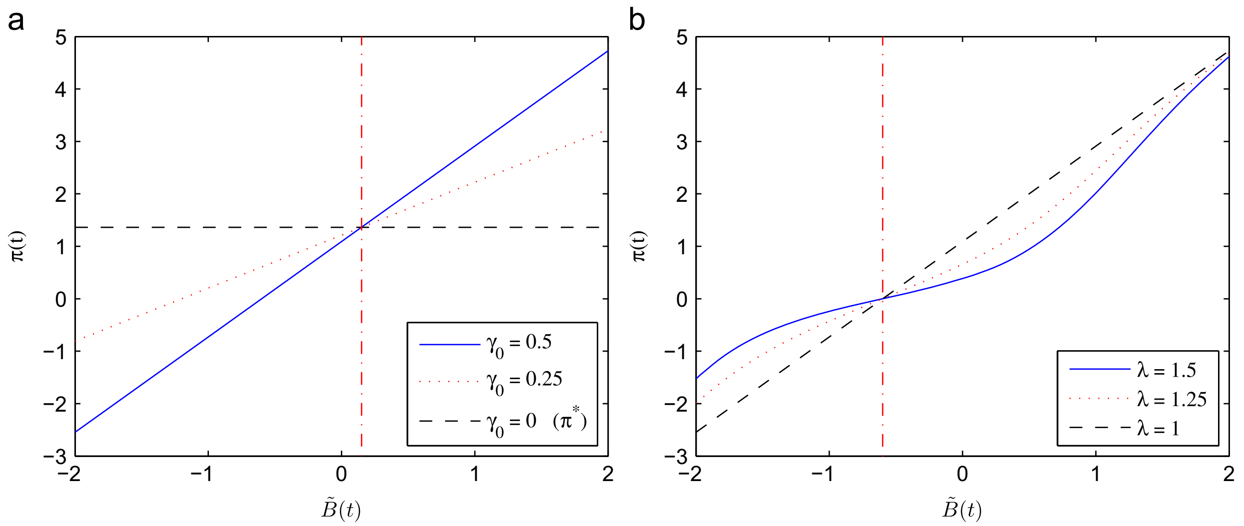

Fig. 1. The effects of γ_0 (a) and λ (b) on optimal portfolio $\pi(t)$ (Parameter values: $\overline{W}=W_0=1$, r_f=0.0408, $\theta=0.3$, $\sigma=0.22$, t=0.5, T=1, and b=0): (a) the effect of γ_0 ($\lambda=1$), (b) the effect of λ ($\gamma_0=0.5$).

where

$$d_1(x)=-\sqrt{\frac{2\ln g(T)-2\ln x}{\frac{\gamma_0 T+1}{\gamma_0}}}-\theta_m(t),\quad d_2(x)=\sqrt{\frac{2\ln g(T)-2\ln x}{\frac{\gamma_0 T+1}{\gamma_0}}}-\theta_m(t),$$

$$d_3(x)=-\sqrt{\frac{2\ln g(T)-2\ln x}{\frac{\gamma_0 T+1}{\gamma_0}}}-\frac{\gamma_0 t+1}{\gamma_0 T+1}\theta_m(t),\quad d_4(x)=\sqrt{\frac{2\ln g(T)-2\ln x}{\frac{\gamma_0 T+1}{\gamma_0}}}-\frac{\gamma_0 t+1}{\gamma_0 T+1}\theta_m(t),$$

$$d(t)=\sqrt{\frac{\gamma_0^2(T-t)}{(\gamma_0 t+1)(\gamma_0 T+1)}},\quad \overline{d}(t)=d(t)\sqrt{\frac{\gamma_0 t+1}{\gamma_0 T+1}}.$$

Fig. 1 presents the effects of incomplete information, preference bias and belief bias on $\pi(t)$. Examining this figure enables us to reveal the dependence of the investor's behavior on the parameters γ_0, λ, and b. Fig. 1a investigates the effects of information incompleteness γ_0 on $\pi(t)$ when there is neither preference bias nor belief bias (i.e., $b=0,\lambda=1$). It is clear from the figure that as γ_0 increases, the sensitivity of $\pi(t)$ to the variation in $\tilde{B}(t)$ also increases. This induces a greater deviation of $\pi(t)$ from the "objective" best strategy (i.e., which corresponds to the true parameter value and leads to the highest expected return in the ex post sense),

$$\pi^*=\frac{\mu-r_f}{\sigma^2},$$

and therefor causes a lower level of expected (objective) return.

Fig. 1b presents the effects of λ on $\pi(t)$ in the case of b=0 and $\gamma_0=0.5$. The wedges among the curves illustrate that increasing λ decreases the size of the optimal risky proportion $\pi(t)$, and produces a greater deviation from the loss-neutral (LN) case ($\lambda=1$). Note that the effect of loss aversion observed in Fig. 1b and the effect of information incompleteness observed in Fig. 1a can be opposite in some situations. A natural question then is whether loss aversion can be advantageous for smoothing away the negative effect brought in by incomplete information?

Based on the patterns revealed in Fig. 1, Fig. 2 further investigates the effects of loss aversion on the optimal asset allocation when b=0. The figure illustrates that we may characterize three subregions in the $\tilde{B}(t)$ space. In the "good-state" region [$\tilde{B}(t)\geq\overline{B}$] and the "bad-state" region [$\tilde{B}(t)<\underline{B}$], the effect of loss aversion goes in the opposite direction with the effect of information incompleteness, where loss aversion reduces the deviation from the investor's optimal strategy to π^*.[4] This effect makes the investor's strategy fit π^* better, and thus tends to increase the investor's objective expected return. We refer this beneficial effect as the *desirable loss aversion effect*. In the intermediate region [$\underline{B}\leq\tilde{B}(t)<\overline{B}$], the effect of loss aversion goes in the same direction with the effect of information incompleteness, where loss aversion

[4] The optimal growth strategy π^* is used as the benchmark for evaluating the relative performance of various strategies because it leads to the highest (objective) expected return in the market.

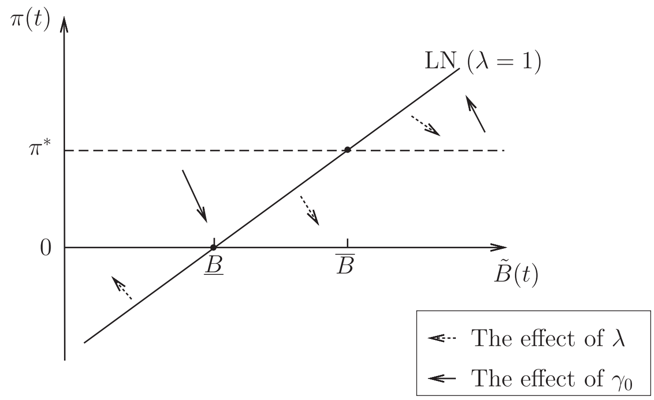

Fig. 2. The effects of loss aversion and incomplete information when $b=0$.

magnifies the deviation between the investor's optimal strategy and π^*. This effect tends to decrease the investor's objective return on average. We refer this hazardous effect as the *excessive loss aversion effect*.

The final effect of loss aversion must be the net effect of these two opposing tendencies. In the complete-information case with $\gamma_0 = 0$, as the LN investor's optimal strategy coincides with π^*, there only exists the intermediate region, and hence the excessive loss aversion effect. When γ_0 is positive, however, both the good-state and bad-state subregions (and thus the desirable effect of loss aversion) emerge. By some calculations, we can further obtain[5]

$$\underline{B} = -\frac{\theta}{\gamma_0}, \quad \overline{B} = \theta t. \tag{9}$$

Consequently, as γ_0 increases, the bad-state subregion grows at the expense of the intermediate subregion, which implies that a greater information incompleteness will both reduce the chance of the excessive loss aversion effect and enhance the chance of the desirable loss aversion effect. As a consequence, although the excessive loss aversion effect dominates when information is rather complete, the desirable loss aversion effect gradually takes over as information becomes sparser. In this way, when the desirable loss aversion effect becomes prominent enough, we would expect that loss aversion can finally offer a beneficial decision-making service. As such, incomplete information leaves "rooms" for loss aversion to improve the investor's decisions.

The above analysis should not be viewed as being specific to the particular settings of our model. In the literature, Merton (1971) and Williams (1977) have shown that incomplete information creates a new state variable. As the true stock return process is unaffected by this state variable, information incompleteness turns out to be equivalent to a source of completely unpriced or unrewarded risks. By taking more precautions against losses, loss aversion then works more effectively in dealing with unrewarded risks than rewarded risks. Specifically, its effect on unrewarded risks tends to increase the actual expected value, while its effect on rewarded risks tends to decrease the actual expected value. Thus, although the latter effect dominates when the source of unrewarded risks, information incompleteness, is negligible, the former effect gradually takes over when the perceived information becomes sparser. Consequently, loss aversion, if appropriate, can be well justified by incomplete information in the sense of both loss avoidance and value creation, and hence may arise as a reasonable psychological response to the threat of information uncertainty.

On the other hand, this potential improvement caused by loss aversion could be further enhanced once the investor further incorporates the belief bias (b) properly. Here, optimism ($b > 0$) is required as a "rosy view" leads investors to pursue a higher risky position and therefor can counterbalance the underinvestment effects of loss aversion. In this way, loss aversion and optimism, together, can actually improve the investor's portfolio decisions in some incomplete-information cases, and therefor could be viewed as the optimal responses to incomplete information.

4. Loss aversion, optimism, and expected return

We now present a systematic analysis of the adaptation problem (P1′).[6]

4.1. Higher returns or psychological biases?

At first, we examine the two primary implications from Section 3: (1) When information is incomplete, optimism is hazardous for LN investors, but may be beneficial for loss-averse investors; (2) Loss aversion is harmful when information

[5] For the case of $\lambda = 1$ and $b=0$, we can obtain the investor's optimal strategy as $\pi(t) = (\theta + \gamma_0 \tilde{B}(t))/[\sigma(\gamma_0 t + 1)]$.
[6] For the market where investors operate, we adopt the parameters estimated by Ang et al. (2005) based on U.S. equity returns from 1926 to 1998: $r_f = 0.0408$, $\theta = 0.30$, and $\sigma = 0.22$. Our basic results are robust to the values of these parameters.

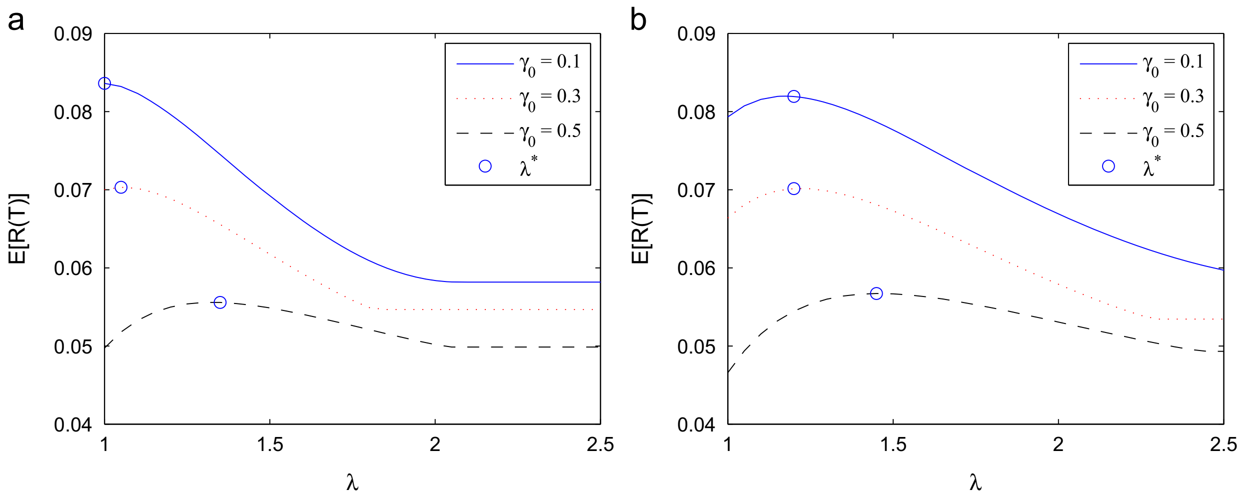

Fig. 3. The actual expected return against λ (Parameter values: $W_0=1$, r_f=0.0408, $\theta=0.3$, $\sigma=0.22$, T=1, and $\overline{W}=W_0$): (a) b=0 and (b) b=0.2.

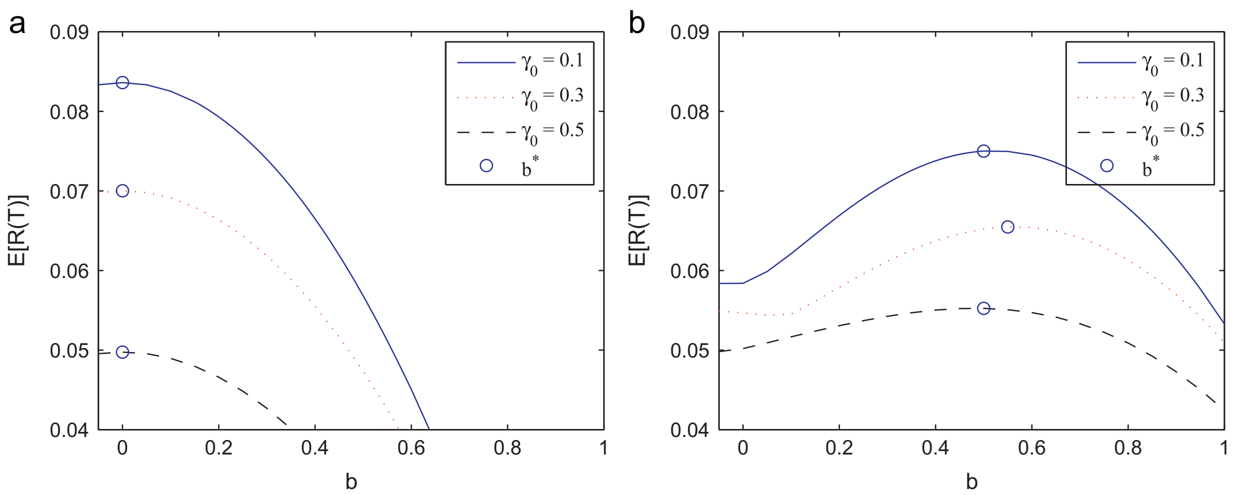

Fig. 4. The actual expected return against b (Parameter values: $W_0=1$, r_f=0.0408, $\theta=0.3$, $\sigma=0.22$, T=1, and $\overline{W}=W_0$): (a) $\lambda=1$ and (b) $\lambda=2$.

is roughly complete, but can be advantageous when information becomes sufficiently sparse. To address the issues, we then examine the relationships between the actual expected return (E[$R(T)$]) and different parameters based on Corollary 1.

Fig. 3 plots the investor's expected return as a function of λ for three different levels of γ_0. Fig. 3a presents the investor's expected return in the case b=0. Evidenced from the figure, the curve with $\gamma_0=0.1$ is monotonically decreasing, while the curves with $\gamma_0=0.3$ and 0.5 are single-peaked. These results are consistent with the intuitions described above.

Fig. 3b further presents the expected return of an investor with b=0.2, i.e., an optimistic case. All three curves in the figure show a single peaked pattern with a peak around $\lambda=1.3$. We can see that loss aversion is the optimal preference feature of this optimistic investor, even when information is rather complete.

Fig. 4 plots the investor's expected return as a function of b for three different levels of γ_0. Fig. 4a presents the expected returns of the LN investor ($\lambda=1$). The figure shows that all three curves reach their peaks at b=0, which implies that optimism (or pessimism) alone is useless for enhancing the expected return. That said, optimism (or pessimism) should not arise in the belief of an LN investor, thus confirming the results in the end of Section 3.1.

Fig. 4b plots the expected return of a loss-averse investors ($\lambda=2$) against belief bias b, and shows a single-peaked pattern with a peak around b=0.5. It turns out that the optimal belief of a loss-averse investor is positively biased.

Taking together, we would expect that there exists a certain critical value $\overline{\gamma}$, and the condition

$$\gamma_0 > \overline{\gamma} \tag{10}$$

ensures that loss aversion can be advantageous for the investor. Meanwhile, consistent with the motto "Contraria sunt Complementa" (opposites are complementary), loss aversion and optimism demonstrate to be complementary for making good decisions.

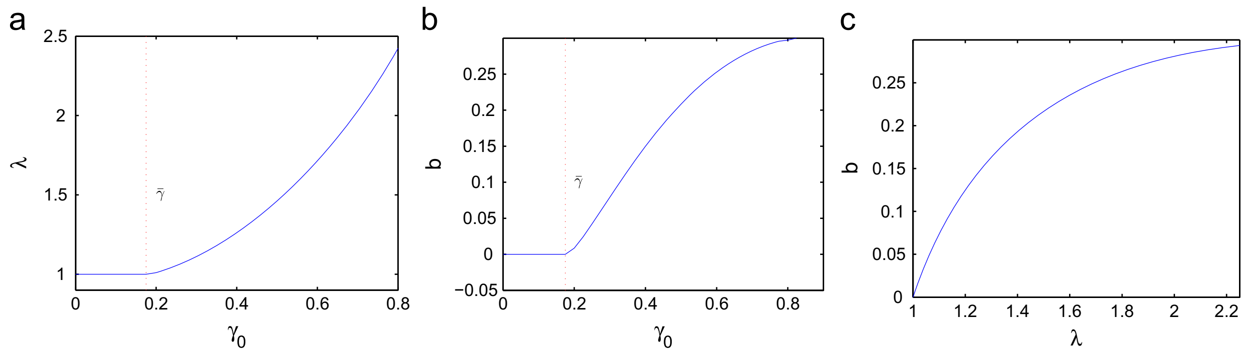

Fig. 5. Optimal (λ, b) against γ_0 (Parameter values: $W_0 = 1$, r_f=0.0408, $\theta = 0.3$, $\sigma = 0.22$, T=1, and $\overline{W} = W_0$).

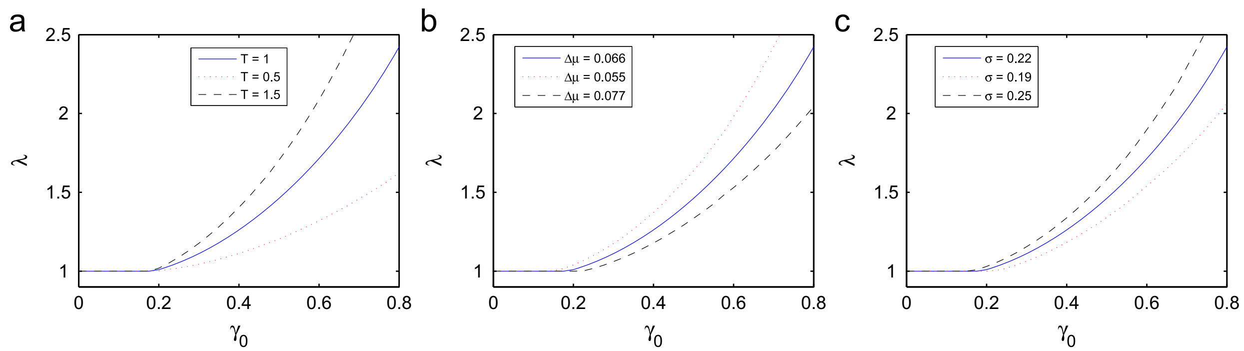

Fig. 6. Sensitivity analysis of optimal λ (Default parameter values: $W_0 = 1$, r_f=0.0408, $\sigma = 0.22$, $\Delta\mu = 0.066$, T=1, and $\overline{W} = W_0$): (a) The effect of time horizon; (b) the effect of risk premium; and (c) the effect of volatility.

4.2. How do optimal psychological features change with incomplete information?

Our results up to this point have suggested that information incompleteness can make rooms for loss aversion and optimism. The next step then is to characterize more precisely how optimal psychological biases relate to information incompleteness.

Fig. 5 displays the relationships among λ, b, and γ_0 obtained from solving the problem (P1′). We can see that there exists some critical level $\overline{\gamma}$ in the figure, as indicated previously. We may use this level to classify the situations into two subsets. Denote (λ^*, b^*) as the solution of the psychological adaptation problem (P1′). In the subset $\gamma_0 \le \overline{\gamma}$, the unbiased alignment $(\lambda^* = 1, b^* = 0)$ prevails. In the subset $\gamma_0 > \overline{\gamma}$ where available information is not precise enough for the investor, (i) the investor's best attitudes are characterized by both loss aversion and optimism, i.e., $\lambda^* > 1, b^* > 0$; and (ii) the investor's best attitudes become more loss-averse and more optimistic when the information becomes further sparser, i.e., both λ^* and b^* increase with γ_0. These results are consistent with the intuition we discussed previously, and demonstrate again the role of incomplete information in determining loss aversion and optimism.

4.3. Comparative statics analysis

Now, we have reconciled two biases, loss aversion and optimism, with functional and adaptive arguments. How sensitive are these results to alternative specifications? The investment problem in our model is determined by the horizon parameter T, the equity risk premium parameter $\Delta\mu = \mu - r_f$, and the volatility parameter σ. We now examine their effects on the endogenous λ^*.[7]

Fig. 6 displays the sensitivity analysis of λ^* to the parameter T, $\Delta\mu$, and σ. In the figure, for every considered case, we can clearly see that loss aversion emerges when γ_0 attains certain level $\overline{\gamma}$, and then grows when γ_0 further increases. This invariant result highlights that our previous conclusion is robust to a wide range of market (or asset) specifications.

The figure also exhibits some new features in the region $\gamma_0 > \overline{\gamma}$. In Fig. 6a and b, the wedges among curves illustrate that an increase in T, or an decrease in $\Delta\mu$, induces an increase in λ^*. That said, loss aversion tends to be more prominent for a longer evaluation period, or a lower risk premium. Fig. 6c further shows that λ^* increases with σ in the region $\gamma_0 > \overline{\gamma}$. In other words, we would expect that loss aversion becomes more prominent in a more volatile situation.

[7] The results about b^* are similar.

5. Discussion

We have established the relationships among loss aversion, optimism and incomplete information. These predictions are generic in that they rely only on the most natural assumption in economics and finance, value maximization, and also in that all built-in elements in the model (loss aversion, optimism and incomplete information) are prevalent phenomena in individual decision making. In this sense, our results are applicable across investment setting types. Likewise, it is not surprising that these relationships are also qualitatively well in line with two general tendencies of psychological bias evolvement, which Hirshleifer (2001) summarized one as: "Many (though not all) of the cognitive biases are stronger for individuals with low cognitive ability or skills than for those with high ability or skills," and the other as: "People are likely to be more prone to bias in valuing securities for which information is sparse."

Then it is useful to describe how our results relate to the behavioral finance literature in more details. On the empirical side, we would want to be able to show that our bounded rationality approach can be useful for structuring analysis of the observed anomalies in individual investor behavior. We then present a brief investigation by focusing on loss aversion. It is well documented in the literature (see, e.g., Daniel et al., 2002; Barberis and Thaler, 2003; Barber and Odean, 2011) that, across countries worldwide, individual investors tend to exhibit loss-averse behavior, e.g., the disposition effect.[8] In a recent overview of the stock trading behavior of individual investors, Barber and Odean (2011) further summarize several general aspects of the disposition effect: it is most pronounced for financially unsophisticated investors or hard-to-value stocks; meanwhile, trading experience tends to diminish it over time. Our model explains loss aversion as a feature related to information incompleteness. Clearly, experience, sophistication, stock-level uncertainty, and so on, all directly determine the extent of information incompleteness. These stylized facts of individual trading behavior are thus certainly consistent with the simple mechanism invoked by our model. In this sense, the large body of empirical and experimental evidence about the disposition effect seems less puzzling once we recognize the fact that incomplete information is all-pervading in individual decision making.

The validity of our model for illuminating the financial literature on the disposition effect can be further confirmed by investigating the link between loss aversion and the observed disposition effect in a rigorous model. For example, Yao and Li (2011) formally demonstrate that loss aversion has a positive causal effect on the investor's negative-feedback trading propensity, which is consistent with the disposition effect, when prospect theory's value function is used to cope with the deviations of actual investor behavior from standard rationality assumptions. In that study, we also present an empirical work on the disposition effect by calibrating the continuous-time portfolio choice model under prospect theory and incomplete information to match the trading characteristics reported by Odean (1998).[9] We find that by incorporating implied information incompleteness (similar to implied volatility in option pricing), prospect theory is more ready to predict the disposition effect. The results show that the portfolio model fits the data well. The calibration also produces reasonable levels of implied information incompleteness, where $\gamma_0 \approx 0.1$ for a wide range of λ's values, and plausible relationships that the disposition effect tends to be weaker when information becomes more precise, or the loss aversion coefficient decreases.

On the theoretical side, our results may prove valuable for integrating various financial theories. In the financial literature, the concepts of incomplete information, loss aversion, and optimism are usually presented independently as competing ways to explain some well-known financial anomalies (e.g., Brav and Heaton, 2002). By establishing their relationships, this paper then takes a further step in the direction of integrating these concepts together, which is at the heart of our growing understanding of how, and to what extent, rational and behavioral theories can be combined. For example, the *limits of arbitrage* problem, which is due to incomplete information in the study of Shleifer and Vishny (1997), could be further exacerbated if endogenous loss aversion is also considered.

A few final remarks are in order. First, some topics, as those often encountered in establishing an evolutionary rationale in economics (e.g., Robson, 2002), remain to be explored. For example, how exactly the psychological adaptation process operates in practice is not obvious as the information incompleteness perceived by individuals is largely unobservable. Additional research about this question in greater details could prove necessary and beneficial.

Second, our results should not be exaggerated. We do not claim that the adaptive interpretation of bounded rationality is descriptive of every conceivable behavior. But those adaptive psychological features are of more relevance for financial and other economists because, e.g., the features are more likely to be engaged in asset pricing.

Third, there could be many contributing factors about loss aversion and optimism other than information incompleteness. We do not intend to deny behavioral explanations based on individual irrationality. Rather, we argue that such irrational explanations may not be always necessary. As long as we acknowledge that incomplete information is an important determinant of individual investor behavior, the link between loss aversion (optimism) and investment mistakes should not be held as a self-evident truth. Kahneman has emphasized in his Nobel address: "I am now quick to reject any description of our work as demonstrating human irrationality."[10] The work in this paper reinforces his argument in the context of investment decisions.

[8] This trading pattern suggests that investors are more prone to realizing gains than losses. Also, there is mounting evidence of short-term contrarian behavior, buying after prices decrease and selling after prices increase.

[9] The estimates in Odean (1998) are obtained based on the stock-trading behavior of 10,000 households with accounts at a large discount brokerage firm.

[10] See http://nobelprize.org/economics/laureates/2002 for Kahneman's Nobel address.

6. Conclusion

Based on Simon's "scissors" analogy, this paper explicitly models how bounded rationality can contribute to the origin and structure of loss aversion and optimism. We show that loss aversion and optimism can be beneficial in terms of increasing the true mean growth rate of wealth accumulation when information is sparse. Rather than expressing loss aversion and optimism as "reasoning errors relevant to economic decisions" (Conlisk, 1996), the results emphasize that the psychological biases can be functional regarding our information-processing limitations. As such, for bounded rational agents, decision-making under uncertainty may naturally occur in a way reflecting loss aversion and optimism. This bounded-rationality mechanism offers an interesting way to link different theories together within the value-maximization hypothesis, and hence may be a promising line of inquiry for our continued efforts to facilitate a systematic accumulation of financial knowledge from different sources toward an internally consistent big picture.

Acknowledgements

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Appendix A

Proof of Proposition 1. We can solve the static problem by following the steps of the method implemented by Basak and Shapiro (2001) and Berkelaar et al. (2004). Applying the Lagrangian method, problem (P2′) can be further written as

$$\max_{W \geq 0} U(W) - y\zeta(T)W = \begin{cases} \ln W - \ln \overline{W} - y\zeta(T)W & \text{if } W \geq \overline{W}, \\ \lambda(\ln W - \ln \overline{W}) - y\zeta(T)W & \text{if } W \leq \overline{W}, \end{cases}$$

where $y \geq 0$ is the Lagrangian multiplier.

For $W \geq \overline{W}$, we can obtain the following local maximum:

$$W_1 = \begin{cases} \dfrac{1}{y\zeta(T)} & \text{if } \zeta(T) \leq \underline{\zeta}, \\ \overline{W} & \text{if } \zeta(T) > \underline{\zeta}, \end{cases}$$

where $\underline{\zeta} = 1/(y\overline{W})$.

For $W \leq \overline{W}$, we get the local maximum as

$$W_2 = \begin{cases} \overline{W} & \text{if } \zeta(T) \leq \overline{\zeta}, \\ \dfrac{\lambda}{y\zeta(T)} & \text{if } \zeta(T) > \overline{\zeta}, \end{cases}$$

where $\overline{\zeta} = \lambda/(y\overline{W})$.

We then compare the local maxima, W_1 and W_2, to determine the global maximum. If

$$f(\zeta(T)) = U(W_1) - y\zeta(T)W_1 - [U(W_2) - y\zeta(T)W_2] \geq 0,$$

the global optimal solution is W_1; otherwise, it is W_2. For $\underline{\zeta} \leq \zeta(T) \leq \overline{\zeta}$, we have $W_1 = W_2 = \overline{W}$; so the optimal wealth in this case is $\overline{W}$.

By some calculations, we can derive the explicit expression of the optimal terminal wealth as follows:

$$W(T) = \begin{cases} \dfrac{1}{y\zeta(T)} & \text{if } \zeta(T) < \underline{\zeta}, \\ \overline{W} & \text{if } \underline{\zeta} \leq \zeta(T) \leq \overline{\zeta}, \\ \dfrac{\lambda}{y\zeta(T)} & \text{if } \zeta(T) > \overline{\zeta}. \end{cases} \tag{A.1}$$

As $\zeta(T) \in [0, g(T)]$, we can further simplify the optimal wealth expression (A.1) into a more compact formula. When $\lambda/(y\overline{W}) \geq g(T) > W_0/\overline{W}$, the optimal terminal wealth can be simplified to

$$W(T) = \begin{cases} \dfrac{1}{y\zeta(T)} & \text{if } \zeta(T) < \underline{\zeta}, \\ \overline{W} & \text{if } \zeta(T) > \underline{\zeta}. \end{cases} \tag{A.2}$$

When $W_0/\overline{W} \geq g(T)$, the optimal terminal wealth can be written as

$$W(T) = \frac{1}{y\zeta(T)}, \tag{A.3}$$

where $y = 1/W_0$.

Note that the formulation in (A.2) corresponds to the case of portfolio insurance, in which the terminal wealth is confined to be above the reference level $\overline{W}$ and all unfavorable states are insured (see, e.g., Basak, 1995; Grossman and Zhou, 1996). The formulation in (A.3) further reduces to the loss neutral, i.e., log return, case. In these two special cases, there exists no direct effect of loss aversion on the investor's choice. □

Proof of Corollary 1. It is clear from (8) that the investor's (log) return is given as

$$R(T|\lambda,b) = \begin{cases} -\ln y - \ln \zeta(T) & \text{if} -\ln \zeta(T) > \ln y + \ln \overline{W}, \\ \ln \overline{W} & \text{if} -\ln \lambda + \ln y + \ln \overline{W} \leq -\ln \zeta(T) \leq \ln y + \ln \overline{W}, \\ \ln \lambda - \ln y - \ln \zeta(T) & \text{if} -\ln \zeta(T) < -\ln \lambda + \ln y + \ln \overline{W}. \end{cases}$$

When $\lambda = 1$, we have $y = W_0 = 1$ and can hence write the return function as[11]

$$R(T|1,b) = -\ln \zeta(T).$$

Given the true value of μ, we have

$$\tilde{B}(T) \sim N(\theta T, T). \tag{A.4}$$

Based on (3) and (A.4), we can derive the objective distribution of $\theta_m(T)$ as follows:

$$\theta_m(T) \sim N\left(\frac{\theta_0 + \gamma_0 \theta T}{\gamma_0 T + 1}, \frac{\gamma_0^2 T}{(\gamma_0 T + 1)^2}\right).$$

Note that this distribution is different from the subjective distribution of $\theta_m(T)$. Some calculations then yield the expression of the actual expected return of the investor's optimal choice. □

Proof of Proposition 2. It is easy to show by Itô's rule that $\zeta(t)W(t)$ is a martingale with respect to the price filtration (see, e.g., Lakner, 1998)

$$W(t) = \mathrm{E}\left[\frac{\zeta(T)}{\zeta(t)} W(T) \middle| \mathcal{F}_t^S\right]. \tag{A.5}$$

We can rewrite Eq. (2) as

$$d\theta_m(t) = \frac{\gamma_0}{\gamma_0 t + 1} d\overline{B}(t).$$

Then, by Itô lemma, we can obtain the subjective distribution of $\theta(T)$ as

$$\theta_m(T) \sim N\left(\theta_m(t), \frac{\gamma_0^2 (T-t)}{(\gamma_0 t + 1)(\gamma_0 T + 1)}\right), \tag{A.6}$$

where $N(\mu,\sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Substituting the above expression and the terminal wealth distribution into (A.5), we can obtain the optimal wealth processes in Proposition 2.

Using Itô lemma again, we know that the stochastic term in $dW(t)$ is

$$\frac{\partial W(t)}{\partial \theta_m(t)} \frac{\gamma_0}{\gamma_0 t + 1} d\overline{B}(t).$$

Comparing the above term with the stochastic term in (4), we obtain

$$\pi(t) = \frac{1}{W(t)\sigma} \frac{\gamma_0}{\gamma_0 t + 1} \frac{\partial W(t)}{\partial \theta_m(t)}.$$

Substituting the optimal wealth processes into the above equation leads to the optimal strategies in Proposition 2.

For the two special cases in the Proof of Proposition 1, we can also get the corresponding results as follows. If $\lambda/(y\overline{W}) \geq g(T) > W_0/\overline{W}$, the time-$t$ optimal wealth is given by

$$W(t) = \frac{1}{y\zeta(t)} \left[\Phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right) + 1 - \Phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right] + \overline{W} e^{-r_f(T-t)} \left[\Phi\left(\frac{d_4(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T + 1}\sqrt{T-t}}\right) - \Phi\left(\frac{d_3(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T + 1}\sqrt{T-t}}\right)\right].$$

[11] Lakner (1998) has obtained that the optimal terminal wealth of a log-utility investor is $W(T) = W_0/\zeta(T)$, and the corresponding optimal trading strategy is $\pi(t) = (m_0 - r_f)/\sigma^2$.

The fraction of wealth invested in the stock is

$$\pi(t)=\frac{1}{W(t)\sigma}\left\{\frac{\theta_m(t)}{y\zeta(t)}\left[\Phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right)+1-\Phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right]-\frac{1}{yd(t)\zeta(t)}\frac{\gamma_0}{\gamma_0 t+1}\left[\phi\left(\frac{d_1(\underline{\zeta})}{d(t)}\right)-\phi\left(\frac{d_2(\underline{\zeta})}{d(t)}\right)\right]\right.$$
$$\left.-\frac{\overline{W}}{\sqrt{T-t}}e^{-r_f(T-t)}\left[\phi\left(\frac{d_4(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)-\phi\left(\frac{d_3(\underline{\zeta})}{\frac{\gamma_0}{\gamma_0 T+1}\sqrt{T-t}}\right)\right]\right\}.$$

If $W_0/\overline{W}\geq g(T)$, the optimal wealth at time t is

$$W(t)=\frac{1}{y\zeta(t)},$$

and the optimal trading strategy is

$$\pi(t)=\frac{m(t)-r_f}{\sigma^2}. \qquad \square$$

References

Ang, A., Bekaert, G., Liu, J., 2005. Why stocks may disappoint. Journal of Financial Economics 76, 471–508.
Barber, B.M., Odean, T., 2011. The behavior of individual investors. Available at SSRN: 〈http://ssrn.com/abstract=1872211〉.
Barberis, N., Thaler, R., 2003. A survey of behavioral finance. In: Constantinides, G., Harris, M., Stulz, R.M. (Eds.), Handbook of the Economics of Finance, Elsevier Science B.V, pp. 1051–1121. (Chapter 18).
Basak, S., 1995. A general equilibrium model of portfolio insurance. Review of Financial Studies 8, 1059–1090.
Basak, S., Shapiro, A., 2001. Value-at-risk-based risk management: optimal policies and asset prices. The Review of Financial Studies 14, 371–405.
Benartzi, S., Thaler, R.H., 1995. Myopic loss aversion and the equity premium puzzle. Quarterly Journal of Economics 110 (1), 73–92.
Berkelaar, A.B., Kouwenberg, R., Post, T., 2004. Optimal portfolio choice under loss aversion. Review of Economics and Statistics 86 (4), 973–987.
Blume, L., Easley, D., 1992. Evolution and market behavior. Journal of Economic Theory 58, 9–40.
Blume, L., Easley, D., 2006. If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets. Econometrica 74 (4), 929–966.
Brav, A., Heaton, J., 2002. Competing theories of financial anomalies. Review of Financial Studies 15 (2), 575–606.
Brennan, M.J., 1998. The role of learning in dynamic portfolio decisions. European Finance Review 1, 295–306.
Brunnermeier, M.K., Parker, J.A., 2005. Optimal expectations. American Economic Review 95 (4), 1092–1118.
Carpenter, J.N., 2000. Does option compensation increase managerial risk appetite? Journal of Finance 55 (5), 2311–2331.
Conlisk, J., 1996. Why bounded rationality? Journal of Economic Literature, 669–700.
Cosmides, L., Tooby, J., 1994. Better than rational: evolutionary psychology and the invisible hand. American Economic Review 84 (2), 327–332.
Daniel, K., Hirshleifer, D., Teoh, S.H., 2002. Investor psychology in capital markets: evidence and policy implications. Journal of Monetary Economics 49 (1), 139–209.
De Long, J.B., Shleifer, A., Summers, L.H., Waldmann, R.J., 1990. Noise trader risk in financial markets. Journal of Political Economy 98 (4), 703–738.
De Long, J.B., Shleifer, A., Summers, L.H., Waldmann, R.J., 1991. The survival of noise traders in financial markets. Journal of Business 64 (1), 1–19.
Dellavigna, S., 2009. Psychology and economics: evidence from the field. Journal of Economic Literature 47 (2), 315–372.
Gennotte, G., 1986. Optimal portfolio choice under incomplete information. Journal of Finance 41, 733–749.
Grossman, S., Zhou, J., 1996. Equilibrium analysis of portfolio insurance. Journal of Finance 51, 1379–1403.
Hakansson, N.H., Ziemba, W.T., 1995. Capital growth theory. Handbooks in Operations Research and Management Science, vol. 9. , Elsevier B.V., pp. 65–86.
Hirshleifer, D., 2001. Investor psychology and asset pricing. Journal of Finance 56 (4), 1533–1597.
Kahneman, D., Tversky, A., 1979. Prospect theory: an analysis of decisions under risk. Econometrica 47, 263–291.
Kelly, J., 1956. A new interpretation of information rate. Bell System Technical Journal 35, 917–926.
Lakner, P., 1995. Utility maximization with partial information. Stochastic Processes and their Applications 56 (2), 247–273.
Lakner, P., 1998. Optimal trading strategy for an investor: the case of partial information. Stochastic Processes and their Applications 76, 77–97.
Liptser, R., Shiryayev, A., 1977. Statistics of Random Processes I. Springer, Berlin.
Lo, A.W., 2005. Reconciling efficient markets with behavioral finance: the adaptive markets hypothesis. Journal of Investment Consulting 7 (2), 21–44.
Merton, R., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373–413.
Merton, R.C., 1980. On estimating the expected return on the market: an exploratory investigation. Journal of Financial Economics 8 (4), 323–361.
Odean, T., 1998. Are investors reluctant to realize their losses? Journal of Finance 53, 1775–1798.
Pástor, L., Veronesi, P., 2009. Learning in financial markets. Annual Review of Financial Economics 1 (1), 361–381.
Robson, A.J., 2002. Evolution and human nature. Journal of Economic Perspectives 16 (2), 89–106.
Rogers, L., 2001. The relaxed investor and parameter uncertainty. Finance and Stochastics 5 (2), 131–154.
Sandroni, A., 2000. Do markets favor agents able to make accurate predictions? Econometrica 68 (6), 1303–1341.
Sharot, T., Riccardi, A.M., Raio, C.M., Phelps, E.A., 2007. Neural mechanisms mediating optimism bias. Nature 450 (7166), 102–105.
Shleifer, A., Vishny, R.W., 1997. The limits of arbitrage. Journal of Finance 52 (1), 35–55.
Simon, H.A., 1957. Models of Man: Social and Rational. John Wiley and Sons, Inc., New York.
Simon, H.A., 1986. Rationality in psychology and economics. Journal of Business 59 (4), S209–S224.
Simon, H.A., 1990. Invariants of human behavior. Annual Review of Psychology 41, 1–20.
Todd, P.M., Gigerenzer, G., 2003. Bounding rationality to the world. Journal of Economic Psychology 24 (2), 143–165.
Tom, S.M., Fox, C.R., Trepel, C., Poldrack, R.A., 2007. The neural basis of loss aversion in decision-making under risk. Science 315 (5811), 515–518.
Williams, J., 1977. Capital asset prices with heterogeneous beliefs. Journal of Financial Economics 5, 219–239.
Yan, H., 2008. Natural selection in financial markets: does it work? Management Science 54 (11), 1935–1950.
Yao, J., Li, D., 2011. Prospect theory and trading patterns. Available at SSRN: 〈http://ssrn.com/abstract=1629766〉.